# Trajectory-Tracking Guidance Law for Low-Thrust Earth-Orbit Transfers

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## Introduction

SOLAR electric propulsion (SEP) may soon become a realistic option for performing Earth-orbit transfers after its successful demonstration by the first new millennium mission, Deep Space 1. Most of the previous work regarding low-thrust Earth-orbit transfers has focused on computing optimal (minimum-fuel or minimum-time) trajectories.[1–3] Trajectory optimization results, however, may not be well suited as a basis for a practical guidance method. For example, a typical transfer from low Earth orbit (LEO) to geosynchronous orbit (GEO) using SEP would require hundreds of days of continuous thrusting and thousands of revolutions. Therefore, it may be impractical to store the optimal control solution onboard a spacecraft. These issues present several challenges for designing a good guidance system for low-thrust Earth-orbit transfers.

Characteristics of a good guidance method include simplicity, autonomy, and near-optimal trajectory performance. An early example of a low-thrust guidance scheme for a lunar mission is presented by Battin,[4] and a recent guidance method for low-thrust interplanetary missions is presented by Noton.[5] The volume of work on guidance methods for many-revolution, low-thrust Earth-orbit transfers is somewhat limited; recent work is presented in Refs. 6–8.

This Note presents a guidance scheme for performing low-thrust Earth-orbit transfers. The guidance computes the thrust steering controls so that a stored reference trajectory is tracked. A numerical simulation of a low-thrust LEO-GEO transfer is presented in order to demonstrate the proposed guidance scheme.

## Trajectory-Tracking Guidance Law

### Thrust Steering Laws

The guidance method computes the thrust steering angles so that the spacecraft tracks a prestored reference trajectory. Our guidance scheme uses two simple control laws and two guidance parameters to define the pitch (in-plane) and yaw (out-of-plane) thrust steering angles. The control law for the pitch steering angle is

$$\sin\alpha_{\text{opt}} = \frac{r\sin\nu}{[r^2\sin^2\nu + 4a^2(e+\cos\nu)^2]^{\frac{1}{2}}}$$

$$\cos\alpha_{\text{opt}} = \frac{2a(e+\cos\nu)}{[r^2\sin^2\nu + 4a^2(e+\cos\nu)^2]^{\frac{1}{2}}} \tag{1}$$

where $r$ is the radial position magnitude, $a$ is the semimajor axis, $e$ is the eccentricity, $\nu$ is the true anomaly, and $\alpha_{\text{opt}}$ is the optimal pitch steering angle for maximum $\mathrm{d}e/\mathrm{d}t$. Pitch angle is measured from the velocity vector to the projection of the thrust vector onto the orbit plane. The optimal control law (1) maximizes the variational equation $\mathrm{d}e/\mathrm{d}t$ and is derived from the first- and second-order conditions for optimality (see Ref. 3 for details). Equation (1) provides maximum $+\mathrm{d}e/\mathrm{d}t$; if a negative eccentricity rate is desired, then a negative sign is included in both numerator terms (i.e., 180 deg is added to $\alpha_{\text{opt}}$). A tradeoff between altering the rates of the in-plane elements ($\mathrm{d}e/\mathrm{d}t$ and $\mathrm{d}a/\mathrm{d}t$) with this pitch steering law is performed by operating on $\alpha_{\text{opt}}$ with a saturation function with limits $\pm\alpha_{\max}$:

$$\alpha = \begin{cases} \alpha_{\text{opt}}, & |\alpha_{\text{opt}}| < \alpha_{\max} \\ \alpha_{\max}\,\mathrm{sgn}(\alpha_{\text{opt}}), & |\alpha_{\text{opt}}| \ge \alpha_{\max} \end{cases} \tag{2}$$

For example, if $\alpha_{\max}$ is set to a small value then the in-plane thrust component is essentially along the velocity vector (i.e., tangent steering and maximum $\mathrm{d}a/\mathrm{d}t$). However, if $\alpha_{\max}$ is set to 180 deg, then the pitch steering maximizes the magnitude of $\mathrm{d}e/\mathrm{d}t$.

The optimal out-of-plane control law for maximum inclination change ($\mathrm{d}i/\mathrm{d}t$) is

$$\beta_{\text{opt}} = (\pi/2)\,\mathrm{sgn}(\cos\theta) \tag{3}$$

where $\theta = \omega + \nu$, $\omega$ is the argument of periapsis, and $\beta_{\text{opt}}$ is the optimal yaw steering angle. Yaw angle is measured from the orbit plane to the thrust vector. Because we wish to simultaneously control in-plane and out-of-plane orbital elements, we replace $\pi/2$ in Eq. (3) with a maximum yaw amplitude angle $\beta_{\max}$. Furthermore, we use the near-optimal yaw steering law

$$\beta = \beta_{\max}\cos\theta \tag{4}$$

so that out-of-plane steering is not wasted when the in-plane longitude angle $\theta$ is near $\pm 90$ deg (this is because $\mathrm{d}i/\mathrm{d}t = 0$ when the spacecraft is $\pm 90$ deg from the nodal crossings regardless of the thrust orientation).

### Predictive Tracking

The guidance problem involves selecting the two steering parameters $\alpha_{\max}$ and $\beta_{\max}$ so that the spacecraft tracks a reference trajectory that is scheduled with orbital energy as the independent variable. Tracking is accomplished by enforcing the constraints

$$\frac{\mathrm{d}\bar{e}}{\mathrm{d}\bar{a}} - \left(\frac{\mathrm{d}e}{\mathrm{d}a}\right)_D = 0, \qquad \frac{\mathrm{d}\bar{i}}{\mathrm{d}\bar{a}} - \left(\frac{\mathrm{d}i}{\mathrm{d}a}\right)_D = 0 \tag{5}$$

where $\mathrm{d}\bar{e}/\mathrm{d}\bar{a}$ and $\mathrm{d}\bar{i}/\mathrm{d}\bar{a}$ are the averaged rates of change (with respect to semimajor axis) for the orbit transfer and $(\mathrm{d}e/\mathrm{d}a)_D$ and $(\mathrm{d}i/\mathrm{d}a)_D$ are the desired rates to be tracked. The averaged time rates of change for the orbital elements $z = [a, e, i]^T$ are

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}t} = \frac{\Delta z}{\Delta t} = \frac{1}{T}\int_{\nu_i}^{\nu_f} \frac{\mathrm{d}z}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\nu}\,\mathrm{d}\nu \tag{6}$$

The integral represents the incremental change in an orbital element during one revolution with all orbital elements held constant except for true anomaly $\nu$. The averaged rate is calculated by dividing the incremental change by the orbital period $T$. Variational equations for $\mathrm{d}z/\mathrm{d}t$ and $\mathrm{d}\nu/\mathrm{d}t$ can be found in Battin.[9] The true anomalies $\nu_i$ and $\nu_f$ represent the exit and entrance into the Earth's shadow. Earth shadow exit and entrance are computed by using the algorithm presented by Neta and Vallado.[10] The averaged rates with respect to semimajor axis are computed by dividing $\mathrm{d}\bar{e}/\mathrm{d}t$ and $\mathrm{d}\bar{i}/\mathrm{d}t$ by $\mathrm{d}\bar{a}/\mathrm{d}t$.

The desired rates to be tracked are computed using the relations

$$\left(\frac{\mathrm{d}e}{\mathrm{d}a}\right)_D = \frac{e^*_{n+1} - \tilde{e}_n}{a^*_{n+1} - a^*_n}, \qquad \left(\frac{\mathrm{d}i}{\mathrm{d}a}\right)_D = \frac{i^*_{n+1} - i_n}{a^*_{n+1} - a^*_n} \tag{7}$$

where the superscript $*$ denotes evaluation on the stored reference trajectory, the tilde indicates mean eccentricity, and subscript $n$ indicates evaluation at the current time. Because eccentricity oscillates at a frequency equal to the orbital period, the mean eccentricity ($\tilde{e}_n$) is computed by averaging samples of eccentricity taken from the preceding orbital revolution (in a real application these data would come from the navigation system; for our results these data come from the numerical simulation of the complete equations of motion). The desired rates $(\mathrm{d}e/\mathrm{d}a)_D$ and $(\mathrm{d}i/\mathrm{d}a)_D$ are computed by linear extrapolation from the current (potentially off-reference) state ($\tilde{e}_n$, $i_n$) to the reference trajectory ($e^*_{n+1}$, $i^*_{n+1}$) at some later energy level $a^*_{n+1}$. It will be shown later that the reference trajectory is also a result of orbital averaging, and therefore the reference states do not contain the oscillatory nature of the true orbital motion. Our guidance scheme represents a predictive tracking method for producing rates that will correct the trajectory back onto the reference path at some later energy level.

The tracking guidance algorithm can now be outlined. At each guidance update the desired predictive tracking rates $(\mathrm{d}e/\mathrm{d}a)_D$ and $(\mathrm{d}i/\mathrm{d}a)_D$ are computed via Eq. (7). The reference elements $e^*_{n+1}$ and

Received 10 June 1999; revision received 7 February 2000; accepted for publication 22 February 2000. 

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$i^*_{n+1}$ are calculated at a future time point (i.e., later semimajor axis $a^*_{n+1}$) by sixth-order polynomial interpolation of the reference trajectory $e^*(a^*)$ and $i^*(a^*)$. The averaged orbital element rates $\mathrm{d}\bar{e}/\mathrm{d}\bar{a}$ and $\mathrm{d}\bar{i}/\mathrm{d}\bar{a}$ are computed using Eq. (6) with steering determined by Eqs. (1), (2), and (4) for an estimated $\alpha_{\max}$ and $\beta_{\max}$. The Earth shadow exit and entrance true anomalies are computed, and the integration over one revolution is computed via Gaussian quadrature. The steering parameters $\alpha_{\max}$ and $\beta_{\max}$ are adjusted by a sequence of one-dimensional Newton iterations. First, the out-of-plane parameter $\beta_{\max}$ is determined by a one-dimensional Newton iteration with a fixed nominal guess for parameter $\alpha_{\max}$. Next, parameter $\alpha_{\max}$ is adjusted using a one-dimensional Newton iteration with the converged value of $\beta_{\max}$ from the preceding step. This process is repeated until Eq. (5) is satisfied with a tolerance of $10^{-8}$ for the $\mathrm{d}e/\mathrm{d}a$ and $\mathrm{d}i/\mathrm{d}a$ errors (all elements are nondimensional with one Earth radii as the reference length). Thrust direction between guidance updates is determined by the converged amplitudes $\alpha_{\max}$ and $\beta_{\max}$ and the respective steering laws. In other words the spacecraft is flown in an open-loop fashion until a new update of the guidance parameters $\alpha_{\max}$ and $\beta_{\max}$ is required.

## Numerical Results

A three-dimensional LEO-GEO transfer is computed for an SEP spacecraft with the following characteristics: input power $P$ is constant at 10 kW, constant specific impulse $I_{sp}$ is 3300 s, and thruster efficiency $\eta$ is 65%. The initial orbit (LEO) is circular with an altitude of 550 km, inclination of 28.5 deg, and ascending node angle of 60 deg. Launch date is set at 1 January 2000 for Earth-shadow computational purposes. The initial spacecraft mass is 1200 kg, and the resulting initial thrust-to-weight ratio is $3.4(10^{-5})$. A standard GEO is the desired target orbit with $a = 42{,}164$ km, $e = 0$, and $i = 0$ deg.

The orbit transfer is simulated by numerically integrating the complete (unaveraged) three-dimensional equations of motion in an Earth-centered Cartesian frame. Numerical integration is performed by a standard fixed-step, fourth-order Runge–Kutta routine. The integration time step is set so that each orbital revolution results in at least 100 integration steps; this was found to be sufficient for acceptable numerical accuracy. Earth-shadow conditions (when $P = 0$) are determined using the geometric relations presented in Ref. 11.

The reference trajectory $[e^*(a^*), i^*(a^*)]$ is the minimum-fuel LEO-GEO orbit transfer as computed by a direct optimization method (see Ref. 3 for details). This direct method (DM) uses orbital averaging for the governing equations of motion, and the reference trajectory consists of 101 data points. The tracking guidance algorithm is used to update the guidance parameters $\alpha_{\max}$ and $\beta_{\max}$ every 1% increase in semimajor axis. Initially, this scheme corresponds to an update interval of about two days; the guidance update interval steadily decreases as the energy rate increases during the orbit transfer. After the semimajor axis of the transfer orbit reaches three Earth radii, the guidance update rate is increased so that $\alpha_{\max}$ and $\beta_{\max}$ are updated every 0.3% increase in semimajor axis. This increased guidance rate improves tracking as the reference trajectory becomes more dynamic and serves to correct for the loss of precision in the orbital averaging approach as the orbital period (semimajor axis) increases. Near the end of the orbit transfer, the guidance update interval is about 8.5 h. All guidance updates to $\alpha_{\max}$ and $\beta_{\max}$ via the sequence of Newton iterations converged in at most 30 iterations.

A successful LEO-GEO transfer was accomplished in a trip time of 207.9 days. The proposed guidance method tracks the reference trajectory nicely, as demonstrated in Fig. 1. The eccentricity history of the guided trajectory oscillates about the reference profile (each oscillation represents an orbital revolution). The guidance method maintains good tracking performance despite this oscillatory behavior. Aside from the oscillatory nature in eccentricity, it is difficult to distinguish the guided trajectory histories from the reference profiles. The simulation is terminated when the spacecraft's energy reaches the GEO energy level. The final values of the three orbital elements at $t_f = 207.9$ days are $a = 42{,}164$ km (because of the simulation termination criteria), $e = 0.0013$, and $i = 0.127$ deg. The final mass of the guided vehicle is 1004.2 kg. The optimal transfer as computed by the DM delivers 1006.2 kg to GEO in 205.4 days. However, the DM uses orbital averaging for the differential equations of motion (with integration step size on the order of two days), whereas the guided transfer presented here uses full numerical integration of the unaveraged dynamical equations with over 500,000 integration steps.

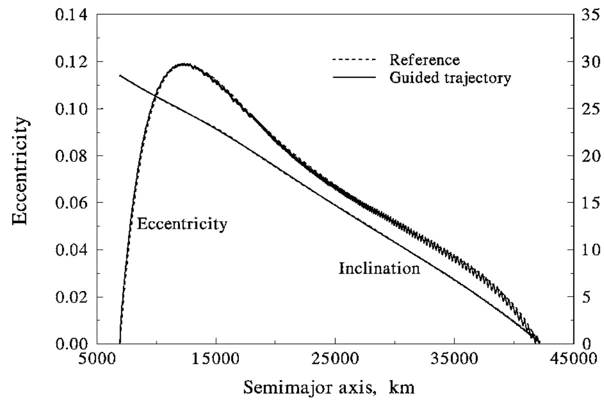


**Fig. 1 Eccentricity and inclination for LEO-GEO transfer.**

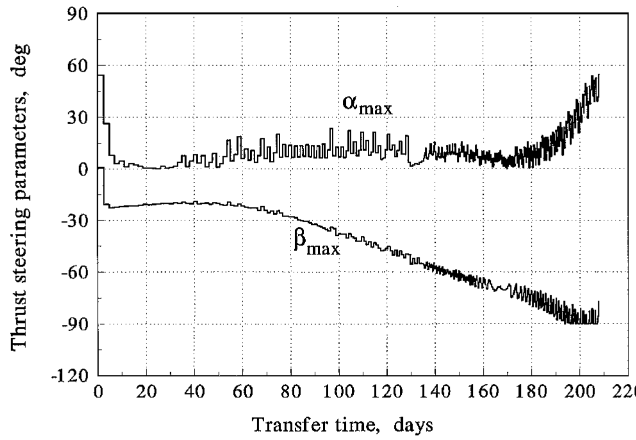


**Fig. 2 Guidance parameters $\alpha_{\max}$ and $\beta_{\max}$ vs time.**

The converged guidance parameters $\alpha_{\max}$ and $\beta_{\max}$ required for the steering laws are presented in Fig. 2. Initially, $\alpha_{\max}$ is fairly large (near 55 deg) so that the steep initial $\mathrm{d}e/\mathrm{d}a$ reference rate is tracked (see Fig. 1). The guidance parameter $\alpha_{\max}$ remains under 30 deg for the majority of the transfer. Therefore, the in-plane thrust component is vectored $\pm 30$ deg from the velocity vector for most of the transfer. The parameter $\alpha_{\max}$ increases at the end of the transfer so that the remaining eccentricity is removed. The magnitude of the yaw angle limit $\beta_{\max}$ steadily increases during the maneuver so that the majority of the plane change is performed at the end of the transfer. The update intervals for the guidance parameters steadily decrease with transfer time.

## Conclusions

A trajectory-tracking guidance law for performing low-thrust Earth-orbit transfers is presented. The guidance uses optimal steering laws, orbital averaging, and a predictive tracking method to track a stored reference trajectory. A LEO-GEO transfer is presented, and the proposed guidance method successfully completes the transfer with near-optimal performance. Because this guidance formulation is simple and efficient, it is well suited for onboard implementation. The guidance method can be augmented with the capability to recompute the reference trajectory; this issue will be addressed in a future paper.

## Acknowledgments

This research was supported by the NASA Graduate Student Researchers Program under Grant NGT8-52850. The authors would like to thank John M. Hanson and Gregory A. Dukeman at NASA Marshall Space Flight Center for their helpful suggestions.

## References

[1] Redding, D. C., and Breakwell, J. V., "Optimal Low-Thrust Transfers to Synchronous Orbit," *Journal of Guidance, Control, and Dynamics*, Vol. 7, No. 2, 1984, pp. 148–155.

[2]Scheel, W. A., and Conway, B. A., "Optimization of Very-Low-Thrust, Many-Revolution Spacecraft Trajectories," *Journal of Guidance, Control, and Dynamics*, Vol. 17, No. 6, 1994, pp. 1185–1192.

[3]Kluever, C. A., and Oleson, S. R., "Direct Approach for Computing Near-Optimal Low-Thrust Earth-Orbit Transfers," *Journal of Spacecraft and Rockets*. Vol. 35, No. 4, 1998, pp. 509–515.

[4]Battin, R. H., *Astronautical Guidance*, McGraw–Hill, New York, 1964, pp. 341–374.

[5]Noton, M., *Spacecraft Navigation and Guidance*, Springer–Verlag, London, 1998, pp. 134–140.

[6]Ilgen, M. R., "Low Thrust OTV Guidance Using Lyapunov Optimal Feedback Control Techniques," American Astronautical Society Paper 93-680, Aug. 1993.

[7]Kluever, C. A., "Low-Thrust Orbit Transfer Guidance Using an Inverse Dynamics Approach," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 1, 1995, pp. 187–189.

[8]Kluever, C. A., "Simple Guidance Scheme for Low-Thrust Orbit Transfers," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 6, 1998, pp. 1015–1017.

[9]Battin, R. H., *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA Education Series, AIAA, New York, 1987, pp. 471–514.

[10]Neta, B., and Vallado, D., "On Satellite Umbra/Penumbra Entry and Exit Positions," *Journal of the Astronautical Sciences*, Vol. 46, No. 1, 1998, pp. 91–103.

[11]Sackett, L. L., Malchow, H. L., and Edelbaum, T. N., "Solar Electric Geocentric Transfer with Attitude Constraints: Analysis," NASA CR-134927, Aug. 1975.

# Long-Range Performance of Suboptimal Periodic Hypersonic Cruise Trajectories

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## Introduction

THE discovery of periodic cruise trajectories for hypersonic flight has evolved from extensive analytical and computational optimization studies performed to determine possible trajectory types that could achieve better fuel consumption savings. With this goal in mind, several researchers have found suboptimal and optimal periodic cruise trajectories that achieve better fuel consumption savings between two destinations over steady-state cruise for a single period. Results from these past studies suggest that fuel consumption savings of 8 to 45% (Refs. 1–6) are possible over a single period. The large difference comes from the use of damped periodic trajectories as compared to purely periodic trajectories.[5] However, to achieve long-range flight, multiple periods will be required. Thus, the question that arises is how does the large performance benefit of damped periodic hypersonic cruise trajectories over a single period vary for multiple periods, as compared to steady-state cruise and purely periodic cruise. The purpose of this Note is to examine this performance by optimizing each period of the trajectory after applying various types of boundary conditions.

Received 3 December 1999; revision received 27 March 2000; accepted for publication 2 April 2000. 

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Table 1 Assumptions for performance comparisons

| Parameterized altitudes | Comments |
|---|---|
| | *Method 1* |
| Eq. (1) | AOA continuity |
| Eq. (2) | Initial and final energy ratios are equivalent |
| | *Method 2* |
| Eq. (1) | AOA continuity |
| Eq. (3) | Initial and final energy ratios are equivalent |
| | *Method 3* |
| Same as method 1 | AOA continuity, initial energy ratio equal to optimal steady-state energy ratio |

## Parameterized Altitude Profiles

Because periodic hypersonic flight trajectories appear to exhibit harmonic behavior, a parameterized form of the altitude profile consisting of only a few harmonics can come quite close to capturing the salient features of the optimal trajectory. Because these harmonics are defined by constants, a suboptimal solution is possible by simply determining the parameters of the altitude profile. This results in less computational effort than a complete optimization of the minimum fuel-consumption rate two-point boundary value problem (TPBVP). The details of this optimization can be found in work done by Chuang and Morimoto[4] and von Eggers Rudd, et al.[5] For multiple periods, an optimization over the complete range of periods would be desirable, but would be complicated and computationally intensive. In the present work, the periods are optimized one at a time and the solutions pieced together. Five methods are evaluated for a variety of parameterized altitude profiles. Each of these methods are described hereafter and summarized in Table 1.

### Method 1

The first method for a long range enforces the states to be continuous at each period's boundary. This produces a realistic flight profile. For steady-state cruise, only Mach number and initial altitude need to be optimized (this steady-state optimization holds for all methods). Periodic hypersonic cruise uses the altitude parameterization given by

$$h = h_a \cos(2\pi/r_f)r + h_b \cos(4\pi/r_f)r + h_c \tag{1}$$

For the first period, the variables to be optimized include the harmonic constants $h_a$, $h_b$, and $h_c$; the initial Mach number $M_0$; throttle on $r_u$; throttle off $r_d$; and final range $r_f$ to minimize the cost function given in Eq. (4). For each period thereafter, only $r_u$, $r_d$, and $r_f$ are optimized so that the initial and final states are always equivalent.

The damped periodic optimization uses an altitude profile as a function of range $r$ given by[5]

$$h = \exp[-\eta(r/r_f)][h_a \cos(2\pi/r_f)r + h_b \cos(4\pi/r_f)r] + h_c \tag{2}$$

Variables to be optimized for the first period include $h_a$, $h_b$, $h_c$, $M_o$, $r_u$, $r_d$, $r_f$, and the damping term $\eta$. Beyond the first period, only $r_u$ and $r_d$ are optimized. The variable $r_f$ must stay constant over multiple periods if continuity of angle of attack (AOA) is desired.

### Method 2

As damped periodic trajectories propagate over long ranges, they will eventually damp out to a mean steady-state cruise altitude. The altitude parameterization given by Eq. (2) enforces the steady-state altitude given by $h_c$. This altitude, however may not be the optimal steady-state altitude. Therefore, a new parameterization is used that allows an exponential damping of the sinusoid trajectory along with a new exponential term on the steady-state altitude variable. This parameterization takes the form

$$h = \exp[-\eta_1(r/r_f)][h_a \cos(2\pi/r_f)r + h_b \cos(4\pi/r_f)r] + \exp[-\eta_2(r/r_f)]h_c + h_d \tag{3}$$

Using this new altitude parameterization, the damped periodic trajectories are optimized like method 1, maintaining continuity of